

Distributed Multi-objective Multidisciplinary Design Optimization algorithms

Amir Noori

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Abstract This work proposes multi-agent systems setting for concurrent engineering system design optimization and gradually paves the way towards examining graph theoretic constructs in the context of multidisciplinary design optimization problem. This paper adapts a coordination strategy based on the well-known nearest neighbor rule and corresponding distributed constrained optimization method. The flow of the algorithm can be described as follow; generated estimates of the optimal (shared design) variables are exchanged locally with neighbor subspaces and then updated by computing a weighted sum of the local and received estimates. To comply with the consistency requirement, the resultant values are projected to local constraint sets. By employing the existing rules and results of the field, it has shown that the dual task of reaching consensus and asymptotic convergence of the algorithms to locally and globally optimal and consistent designs can be achieved. Finally, simulations are provided to illustrate the effectiveness and capability of the presented framework.

Keywords Multidisciplinary design optimization, Consensus algorithms, Projection methods, Distributed optimization

1 INTRODUCTION

Development and optimization of complex engineering systems arise from the challenges of effectively addressing the competing needs of improving performance, reducing costs, and enhancing safety. Modern complex engineering systems are usually heterogeneous, highly interconnected and mutually interdependent, both physically and through a multitude of information and communication networks [Haddad et al., 2006]. Examples of such systems include automobile and rail vehicles design, naval architecture, electronics, computers, and micro-electro-mechanical systems (MEMS), as well as system of systems such as air and ground traffic networks, distributed manufacturing environments, and globally distributed supply networks.

Islamic Azad University, Karaj Branch,
Moazen BLVD, Rajaee-shahr,
P.O. Box 31485-313, Karaj, Iran
Email: amir.noori@kiau.ac.ir

Design and optimization of these complex engineered systems are multidisciplinary in nature, far from optimal, and heavily constrained by both technical and nontechnical considerations. It is therefore impossible for one designer, or even a single design team, to consider the entire problem as a single design problem. Therefore, the design process is decentralized or distributed over a number of design teams that autonomously operates on a single component or aspect of the system. This paper is about Collaborative Multidisciplinary Design Optimization, or CMDO, and deals with concurrent optimization of two or more coupled analysis disciplines with distributed computation models and numerical optimization.

Multidisciplinary design optimization [Alexandrov and Hussaini, 1995] allows designers to incorporate all relevant disciplines simultaneously. The simultaneous optimization is superior to the commonly used sequential design methods, since it can exploit the interactions between the disciplines. Design is both analysis and synthesis, and is compromise in the balance of conflicting requirements. However, finding the best compromise by including all disciplines simultaneously significantly increases the complexity of the problem.

During the past three decades, decomposition-based design optimization strategies [Sobieski and Haftka, 1997] as a natural approach have drawn a great deal of attention of researchers to solve the design problem of complex systems in a distributed way. However, most works in such decentralized design optimization settings address a hierarchical or sequential evaluation of a master problem and some disciplinary sub-problems. The optimal set of design variables for the upper level becomes the objective/constraints responses for the lower level. At each level, disciplines are optimized separately. Accordingly, two general methods are elaborated in literature; single-level methods, which have centralized decision-making authority and do not allow design decisions to be made at the disciplinary level, and multi-level methods, which a central master optimization problem is introduced to coordinate the interactions between the disciplinary sub-problems. In some multi-level methods, such distributed authority for decision making is sometimes referred to as distributed design optimization schemes. However, in this paper, by distributed methods we mean the same decentralized decision making capability without any master or central coordinator.

However, several recently proposed distributed methods are placed in the multi-level category. Typically, these methods hierarchically decompose the underlying design problem into sub-systems along the lines of systems, subsystems, and components or usually partition in a non-hierarchical fashion along disciplinary lines. A review of single-level and multi-level methods can be found in [Cramer et al., 1994] and [Tosserams et al., 2009].

Several partitioning and coordination methods have been proposed for Multidisciplinary Design Optimization (MDO) problem including linear decomposition method (OLD) [Sobieski, 1982], Concurrent Subspace Optimization (CSSO) [Sobieski, 1988], BLISS [Sobieski et al., 2003], analytical target cascading (ATC) [Michelena et al., 1999], Quasiseparable decomposition (QSD) approach [Haftka and Watson, 2005], penalty decomposition formulation [Demiguel and Murray, 2006], Augmented Lagrangian Decomposition method for Quasi-separable problems [Tosserams et al., 2007], and Collaborative Optimization (CO) [Braun, 1996], Enhanced Collaborative Optimization (ECO) [Roth and Kroo, 2008]. A full description of decomposition and coordination strategies in multidisciplinary design optimization problems is beyond the scope of this paper and was already reported in [Tosserams et al., 2009]. As mentioned before, most of these approaches are essentially decentralized in the way that they organize a fully

centralized problem in a hierarchical (i.e., sequential) structure. It is clear that we have reached the limits of what these approaches can do [Allen et al., 2011]. To proceed, we need a more rigorous and deeper understanding of complex engineered systems and how they should be designed.

The penalty relaxation methods [Michelena et al., 1999], [Blouin et al., 2005], [Tosserams et al., 2008]) relax the coupling constraints of MDO problems to arrive at subproblems with separable constraint sets. ALC [Tosserams et al., 2008] provides a flexible coordination structure, not necessarily hierarchical that uses penalty relaxation methods in tandem with algorithms for solving systems of equations. The convergence to KKT points of the original problems is the main advantage of penalty methods. Some penalty relaxations methods have only been developed for quasi-separable problems coupled through a set of coupling variables; coupling objectives and constraints are not allowed. In [Tosserams et al., 2006], an Augmented Lagrangian Relaxation for Analytical Target Cascading using the Alternating Directions Method of Multipliers are proposed. In [Tosserams et al., 2008], a new penalty relaxation coordination method is proposed that can be used to solve MDO problems with coupling variables, a coupling objective, and coupling constraints. However, ALC results in a very large number of consistency constraints; only a subset is actually required to ensure consistency. On the other hand, such methods rely on excessively large penalty factors for sufficiently accurate solutions. Several consistency constraints allocation guidelines have been proposed for ALC implementations in [Allison and Papalambros, 2010].

Our work is also related to game-based design approaches [Hernandez et al., 2002] [Xiao et al., 2005] [Ciucci et al., 2012]. The pioneer work of Lewis et al [Lewis and Mistree, 1997] in the last nineties suggested a game theoretic approach to model interactions in multidisciplinary design. In [Hernandez et al., 2002], the authors investigated a quadratic and eigen-based formulation to enhance convergence speed. They conclude that passing more information generally leads to convergence to a Pareto-optimal set.

This work is built upon the constrained consensus method [Nedic and Ozdaglar, 2009] [Nedic et al., 2010], adapts a distributed computation framework for general complex system design and optimization problem and gradually paves the way towards examining graph theoretic constructs in the context of multidisciplinary design optimization problem. Convergence properties of the agreement protocol can be proved using existing result on algebraic graph theory [Godsil and Royle, 2001], in particular, spectral properties of the underlying graph.

This paper considers the design problem of complex engineering systems, consisting of several (including both local and global) objectives, design variables and constraints corresponding to different disciplines and assigned to several teams. In other words, system-wide disciplines (i.e., those are spread throughout the problem space) are also allowed. In particular, these distributed multi-objective multidisciplinary design optimization problem is presented in a multi-agent setting, providing necessary information passing structure to come up with an appropriate decision. In brief, the flow of the proposed algorithm can be described as follow; generated estimates of shared design variables and optimal linking variables (if any) are exchanged directly among subspaces. Afterwards, each agent computes a weighted sum of its estimates and received estimates. Then, each agent projects the variables received from other agents to its constraint sets to maintain consistency.

Analysis shows that the performance of the algorithm largely depends on the subproblems as well as the network structure and communication protocols. Afterwards, our main focus is on developing distributed algorithms that guarantees asymptotic

consensus on the shared and linking variables while maintains the feasibility with respect to the constraint sets. The proposed multi-agent framework is adapted to flexibly address both system-level (global) and discipline-level (local) issues, without any requirement of objectives and constraints relaxation. Moreover, it has been shown that the proposed framework is in great accordance with the corresponding design and development teams. Finally, the main feature that distinguishes the proposed method from others is its structural flexibility, design autonomy, and rigor mathematical and graphical representation. We believe that the most challenging part of the method (especially for general design optimization problem) is the implementation and analysis of its nonlinear part; i.e., projection algorithm. In addition, coupling between subspaces might be another challenging issue and require further investigations both in theory and in experiment. Attention is focused in this paper on the general case, and some companion papers will be devoted to special cases that allow further analysis.

The rest of the paper is organized as follows. Section II is devoted to provide the necessary mathematical and graph theoretical foundations and computational algorithms used for incremental coordination and projection. Section III introduces the main problem we consider, discusses the assumptions made in the proposed model, and presents our algorithm. Section IV presents the performance analysis and the convergence issues of the algorithm, followed by section V that demonstrates the algorithm and compares the results with that of the well-known All-in-once (AIO) method. Section VI concludes the paper.

2 PRELIMINARIES

In this section, we discuss the standard consensus algorithm and the constrained consensus algorithm. Some of the equations and ideas for this section covering consensus, projection and constrained consensus algorithms, originate from [Nedic et al., 2010], demonstrating the existing strong theoretical foundation. They are included here to ensure a complete statement of the problem at hand and an accurate comparison between coordination strategies.

2.1 Consensus Algorithms

Distributed average consensus algorithms are a class of iterative update schemes that work based on the neighbor interactions. In recent years, there has been a surge of interests in distributed computing methods based on the average consensus algorithms [Jadbabaie et al., 2003][Olfati-Saber and Murray, 2004], and it has found applications in rendezvous, formation control, flocking, attitude alignment, decentralized task assignment, and sensor networks. Let's consider a network of i agents, represented by $V = \{1, 2, \dots, n\}$. The neighbors of node i is a set of nodes $j \in V$ where communicating with node j through a directed link $e = (i, j)$. At each time $k + 1$, we assume that agent i receives information $z_j(k)$ from neighboring agents j and updates its estimate by adding a weighted sum of the local discrepancies, i.e., the differences between neighboring node values and its own. In [Olfati-Saber and Murray, 2004], Olfati-saber and Murray show that the following linear dynamic system:

$$z_i(k+1) = z_i(k) + \sum_{j \in N_i} a_{ij}(k)(z_j(k) - z_i(k)) \quad (1)$$

where $a_{ij}(k)$ is a weight associated with the edge (i, j) , $j \in N_i$ and $k = 0, 1, \dots$, solves a consensus problem. More precisely, let z_k 's be n constants, then with the set of initial states $z_i(0) = z_i$, the state of all agents asymptotically converges to the average value $z = \frac{1}{n} \sum_i z_i$ provided that the network is connected.

2.2 Projection

Let z be an element in a Hilbert space H and let Z be a closed (possibly non-convex) subset of H . We use $P_Z[\bar{z}]$ to denote the projection of a vector \bar{z} onto a closed convex set Z , and define as follows:

$$P_Z[\bar{z}] = \arg \min_{z \in Z} \|\bar{z} - z\|$$

There is always at least one such point for each z , namely where H is a finite dimensional Hilbert space. If Z is convex as well as closed then each z has exactly one projection point $P_Z[z]$ [Luenberger, 1969].

2.3 Constrained Consensus Algorithm

The constrained consensus problem is to achieve asymptotic consensus on the local decision variables, z_i , through information exchange with the neighboring nodes in the presence of the constraint sets, $z_i \in Z_i$. A distributed algorithm for this problem was proposed in [Nedic et al., 2010]. In the algorithm, i^{th} local variable $z_i(k)$ is executed as follows:

$$z_i(k+1) = P_{Z_i} \left[\sum_{j=1}^m a_{ij}(k+1) z_j(k) \right] \quad (2)$$

and it can also be written:

$$v_i(k+1) = z_i(k) + \sum_{j=1}^m a_{ij}(k+1)(z_j(k) - z_i(k)) \quad (3)$$

$$z_i(k+1) = P_{Z_i}[v_i(k+1)] \quad (4)$$

Illustration of the algorithm in 2D case is given in Figure 1. The constrained consensus algorithm has several advantages over alternating projection method, the most significant being that concurrent computation of subspaces is possible. Accordingly, in a distributed setting, internal dynamics of subspaces are coupled together, cooperating (or may be competing) to achieve the overall system objectives:

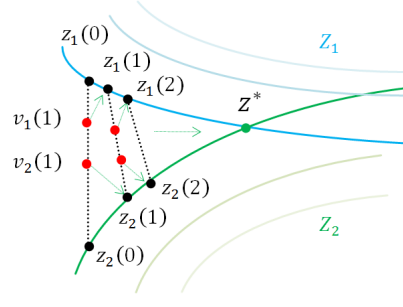


Fig. 1 Constrained consensus algorithm

$$v_i(k+1) = (1 - a_{ij}(k+1))z_i(k) + a_{ij}(k+1)z_j(k) \quad (5)$$

$$z_i(k+1) = P_{Z_i}[v_i(k+1)] \quad (6)$$

where $i \neq j, i, j = 1, 2$.

2.4 Constrained Optimization

Distributed constrained optimization discusses the problem of optimizing the sum of convex objective functions corresponding to m connected agents. The goal of the agents is to cooperatively solve the constrained optimization problem

$$\begin{aligned} \min \quad & \sum_{i=1}^m f_i(z) \\ \text{subject to} \quad & z \in \bigcap_{i=1}^m Z_i \end{aligned} \quad (7)$$

where the local objective function of agent i , $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, is a convex function, and the local constraint set of agent i , $Z_i \subseteq \mathbb{R}^n$, is a closed convex set that are known to agent only. In [Nedic et al., 2010], the following distributed projected subgradient algorithm are proposed for solving problem (7)

$$v_i(k) = \sum_{j=1}^m a_{ij}(k) z_j(k) \quad (8)$$

$$z_i(k+1) = P_{Z_i}[v_i(k) - \alpha_k d_i(k)] \quad (9)$$

where the scalars $a_{ij}(k)$ are nonnegative weights and the scalar $\alpha_k > 0$ is a stepsize. The vector $d_i(k)$ is a subgradient of the agent local objective function $f_i(z)$ at $z = v_i(k)$.

3 DISTRIBUTED MULTI-OBJECTIVE MULTI-DISCIPLINARY DESIGN OPTIMIZATION

In this section we introduce the main problem we consider. We also discuss the assumptions made in our model and propose our algorithm. The general formulation for this section covering multidisciplinary design optimization is in the spirit of the framework outlined in design optimization literature [Alexandrov and Hussaini, 1995].

3.1 MDO Problem Formulation

The optimization problem usually encountered in many engineering system design is considered to be a nonlinear programming problem. The general multidisciplinary design optimization (MDO) problem is formulated as follows:

$$\begin{aligned}
 \min_{z=(z^l, z^s)} \quad & f(x, y, z) \\
 \text{s.t.} \quad & g(x, y, z) \leq 0 \\
 & h(x, y, z) = 0 \\
 & \forall i, j \neq i, y_i = c_{ji}(x_j, y_j, z_j) \\
 & \forall i, x_i = T_i(x_i, y_i, z_i)
 \end{aligned} \tag{10}$$

where

$z = (z_1, z_2, \dots, z_m)$ denote design vector, consist of design variables from different disciplines. Moreover, these variables can be partitioned into shared and local design variables of i th subspace; z_i^s and z_i^l , respectively.

$x = (x_1, x_2, \dots, x_m)$ is state vector and depend on both linking and design variables.

y_i 's denote linking variables and provide coupling among different subsystems.

f, g and h are vector-valued objective function, inequality and equality constraints functions, respectively.

$c_{ji} : (x_i, y_i, z_i) \rightarrow y_j$ denote the coupling function from the subsystem i to the subsystem j .

$T_i : (x_i, y_i, z_i) \rightarrow x_i$ denote state transition functions that compute state variables of subspace i .

We assume that the local objective function f_i and the local coupling c_{ji} and transition T_i functions are known to agent only. Recall that z^* is a local minimum of (10) if there exists $\varepsilon > 0$ such that $f(x, y, z^*) \leq f(x, y, z)$ for all $z \in S \cap B(z^*, \varepsilon)$ and corresponding state and coupling variables, where S is the feasible region.

The optimal design of complex engineering systems involves concurrent optimization of several objectives, constrained by both local and global issues. A distributed variant of equation (10) can be represented as follows:

$$\begin{aligned}
& \min_{z=(z^i, z^s)} [f_0(x, y, z); f_1(x, y, z); \dots; f_m(x, y, z)] \\
& \text{s.t.} \quad g_i(x, y, z) \leq 0 \quad \text{for all } i \\
& \quad \quad h_i(x, y, z) = 0 \quad \text{for all } i \\
& \quad \quad \forall i, j \neq i, y_i = c_{ji}(x_j, y_j, z_j) \\
& \quad \quad \forall i, x_i = T_i(x_i, y_i, z_i)
\end{aligned} \tag{11}$$

This problem can be formulated in a more compact form, which makes it suitable for representation of distributed multi-objective design optimization problems. Let S denote the feasible region of (11), i.e.,

$$\begin{aligned}
S_i = \{ & (x' \in X_i, y' \in Y_i, z' \in Z_i) : \\
& g_i(x', y', z') \leq 0, h_i(x', y', z') = 0 \}, i = 0, 1, \dots, m
\end{aligned}$$

Accordingly, we use φ_i to represent functional evolution of the state and coupling variables of subspace i :

$$\begin{aligned}
\varphi_i = \{ & (x_i \in X_i, y_i \in Y_i, z_i \in Z_i) : x_i = T_i(x_i, y_i, z_i), \\
& y_i = c_{ij}(x_j, y_j, z_j), i \neq j \}
\end{aligned}$$

and then let

$$S = \bigcap_{i=1}^m S_i$$

$$\varphi = \bigcup_{i=1}^m \varphi_i$$

Then the problem (11) can be written:

$$\begin{aligned}
& \min_{z \in Z} F(x, y, z) = [f_0(x, y, z); f_1(x, y, z); \dots; f_m(x, y, z)] \\
& \varphi(x, y, z) \in S
\end{aligned} \tag{12}$$

The goal of the agents is to cooperatively optimize both local and global objectives:

$$f(x) = f_0(x) + \sum_{i=1}^m f_i(x)$$

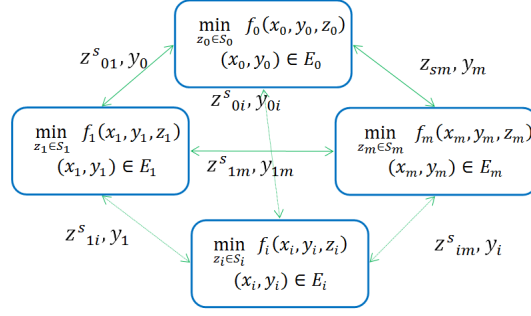


Fig. 2 Digraph represent structure of design optimization problem

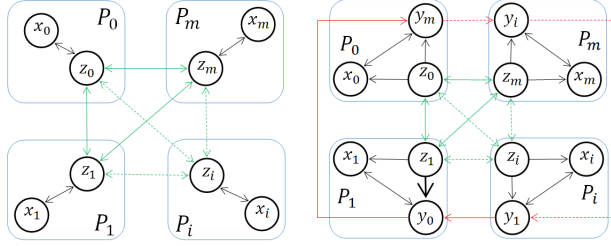


Fig. 3 A typical structure for information-passing; (left) Distributed constrained optimization, (right) Distributed design optimization; shared variables (green), coupling variables (red), and state variables (black)

3.2 Distributed Computation Model

In multiobjective multidisciplinary design problem, interconnections and interactions among subspaces are usually complex. In such complex problems, information-passing among subspaces and structural organization of the subproblems is as important as subspaces optimization. In the following, we discuss the proposed distributed computation model from a graph theoretic perspective to handle information passing among subproblems. Moreover, an analysis section is devoted to discuss basic assumptions and corresponding results from algebraic graph theory, which provides analytical framework, required for analysis of coordination strategy and communication protocols. A detailed discussion of this topic is beyond the scope of this paper and a brief overview of the subject is given. For further information on this topic, readers are referred to [Nedic et al., 2010]. Recall we represent the structure of our design optimization problem, consists of m subproblems by undirected graph $G = (V, E)$. A typical structure of the design problem is illustrated in Figure 2.

In MDO problems, there are at least two different types of information being exchanged; shared design variables and coupled variables. Figure 3 illustrates a typical flow of information among subspaces.

In addition, communication protocols play a critical role in providing effective information sharing between several design teams. A communication protocol consists of a set of rules which govern the orderly exchange of information among entities. Two general classes of protocols are proposed; synchronous and asynchronous protocols. Gossip communication protocol of [Kempe et al., 2003] and the broadcast protocol of

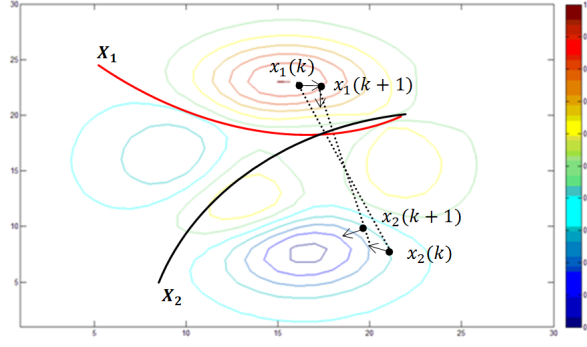


Fig. 4 Illustration of the proposed algorithm

[Aysal et al., 2009] are some examples of these protocols. For a detailed review of this topic see [Aysal et al., 2009].

3.3 Collaborative Multidisciplinary Design Optimization (CMDO) algorithm

We next consider the problem of general multi-objective multi-disciplinary distributed design optimization (M2DO) (11) corresponding to m agents connected over a fixed topology. We also introduce two different implementations of the proposed method for solving this problem. Recall the local design variable associated with subspace i is denoted by $z_i(k)$ and is restricted to lie in a local constraint set Z_i . In the multi-step version of the algorithm, each agent i starts with an initial estimate $z_i(0) \in Z_i$ and update its estimate by combining the estimates received from its neighbors, by taking an optimization step to minimize its objective function. Then she/he exchanges this estimate with neighbor subspaces. Finally, the resulting value is projected into local feasible set. This algorithm illustrates the intuition behind the proposed method and allows employing heuristics in projection and optimization steps. In interleaved version, the optimization step is augmented in the constrained consensus algorithm. We consider this formulation when discussing performance of the algorithm.

Although there are a lot of choices for projection or other steps, but we do not discuss it in this paper. However, the main advantage of this algorithm lies in the fact that it is possible to independently specify each step.

An illustration of the algorithm is presented in Figure 4. In the following sections, we discuss the behavior of this algorithm.

4 PERFORMANCE ANALYSIS

We next discuss the behavior of the proposed algorithm. Analysis of the general MDO problem is difficult because of the coupling among subspaces and internal state transition. In this paper, we focus on some special cases including quadratic objective functions with linear subspaces. Firstly, some existing results in the field are discussed. Then, a class of multidisciplinary design optimization problems are defined and some extra assumptions are placed on information structure and coordination strategy that can guarantee the convergence of the algorithm.

Algorithm 1 Multi-Step Algorithm

Algorithm 1: Multi-step version This algorithm is sequentially executed in three steps as follows:

Update Let's consider the initial estimate of the local and shared design variables $z_i^l(0) \in Z_i^l, z_i^s(0) \in Z_i^s$, respectively. Hereinafter, for the sake of notational simplicity, we omit the super-index s in z_i^s . These estimates z_i , are exchanged directly among neighbor subspaces according to the problem structure. Each agent updates its estimate by forming a convex combination of the local estimate of shared design variable and that of neighbor subspaces. This update mechanism at time t_k is formally stated by the following equation:

$$z_i(k+1) = z_i(k) + \alpha_i(k) \sum_{j=1}^m a_{ij}(k+1)(z_j(k) - z_i(k)) \quad (13)$$

where $a_{ij}(k)$ determines information passing among subspaces. The coupling and state variables are also updated as follow:

$$i, j \neq i, y_i = c_{ji}(x_j, y_j, z_j) \quad (14)$$

$$x_i = X_i(y_i, z_i) \quad (15)$$

At the end of this step, agents share their estimate of shared values and coupling variables among each other.

Optimization Afterwards, each agent sets the updated shared design variable, $z_i(k+1)$ as an initial value, updates his/her objective based on the recent local state variables and received coupling variables and then takes an optimization step according to the following optimization problem $\min_{z_i} f_i(x_i, y_i, z_i)$. Let's denote the resulting optimal design variables at time $t_{(k+1)}$ by $z_i^*(k+1)$.

Projection The last optimization step may result in values outside the feasible region of each subspace. Therefore, we project the resultant value into the feasible region to satisfy constraints

$$z_i(k+1) = P_{S_i}[z_i^*(k+1)] \quad (16)$$

Algorithm 2 Interleaved Algorithm

Algorithm 2: In this algorithm, the optimization step is augmented in the constrained consensus algorithm (2) as follow

$$z_i(k+1) = P_{S_i} \left[\alpha_i(k) \sum_{i=1}^m a_{ij}(k) z_j(k) - \gamma_i(k) d_i(k) \right] \quad (17)$$

Where the vector $d_i(k+1)$ is an optimization step of the objective function of subspace i , i.e., f_i at point $(x_i(k), y_i(k), z_i(k))$, updated at each time step t_k .

4.1 Constrained Consensus

This section provides a summary description of the existing convergence results for distributed optimization. We adopt the following assumptions in our analysis following Blondel et al. [Blondel et al., 2005] and [Nedic et al., 2010].

Assumption1: (Network Topology) We assume information passing among subspaces takes place at discrete time steps, t_k . Recall the weight associated with the

edges in network graph can be denoted by $a_{ij}(k)$, or $[A(k)]_{ij}$, where $A(k)$ is called the *network weight matrix*.

Assumption2: (Weights Rule) There exists a scalar η with $0 < \eta < 1$ such that for all $i, j \in \{1, \dots, m\}$, and $k \geq 0$

1. $a_{ii}(k) \geq \eta$
2. $a_{ij}(k) \geq \eta$ when subspace j communicates with subspace i , and $a_{ij}(k) = 0$ otherwise.
3. $\sum_{i=1}^m a_{ij}(k) = 1$ (row stochastic)

Assumption3: (Connectivity and symmetry) The graph (V, E_∞) is connected, where E_∞ is the set of edges (j, i) representing subspace pairs communicating directly infinitely many times, i.e.,

$$E_\infty = \{(j, i) | (j, i) \in E_k \text{ for infinitely many indices } k\}$$

Moreover, the graph (V, E_∞) is symmetric, i.e., the weights satisfy $a_{ij}(k) = a_{ji}(k)$ for all i, j .

Assumption4: (Bounded intercommunication intervals) If i communicates to j an infinite number of times [that is, if $(i, j) \in E(t)$ infinitely often], then there is some B such that, for all t ,

$$(i, j) \in E(t) \cup E(t+1) \cup \dots \cup E(t+B-1)$$

Assumption5: (Doubly Stochasticity) The vectors

$a_i(k) = (a_{i1}(k), \dots, a_{im}(k))'$ satisfy:

- a) $a_i(k) \geq 0$ and $\sum_{i=1}^m a_{ij}(k) = 1$ for all i and k , i.e., the vector $a_i(k)$.
- b) $\sum_{i=1}^m a_{ij}(k) = 1$ for all j and k .

Remark 1 This assumption establishes that each agent takes a convex combination of its estimate and the estimates of its neighbors. Moreover, second part of this assumption together with Assumption 2, ensures that the estimate of every agent is influenced by the estimates of every other agent with the same frequency in the limit, i.e., all agents are equally influential in the long run [Nedic et al., 2010].

Lemma 1 [Olfati-Saber et al., 2007] *Let G be a connected undirected graph. Then, the algorithm (1) asymptotically solves an average consensus problem for all initial states.*

Remark 2 Simply, this lemma states that if there exist some paths for flow of information, consensus is eventually achieved. Therefore, information exchange among different subspaces with the same shared design variables are necessary for convergence.

Proposition 1 [Nedic et al., 2010] (Consensus) *Let the set $Z = \bigcap_{i=1}^m Z_i$ be nonempty. Also, let Weights Rule, Doubly Stochasticity, Connectivity, and Information Exchange Assumptions hold (cf. Assumptions 2, 3, 4, and 5). For all i , let the sequence $\{z_i(k)\}$ be generated by the constrained consensus algorithm (2). We then have for some $\tilde{z} \in Z$ and all i , $\lim_{k \rightarrow \infty} \|z_i(k) - \tilde{z}\| = 0$.*

4.2 Distributed Constrained Optimization

In this section, we discuss the existing results for the constrained optimization problem. Let's first consider the following assumptions.

Assumption 6: (*Same Constraint Set*)

- The constraint sets Z_i are the same, i.e, $Z_i = Z$ for a closed convex set Z .
- The subgradient sets of each f_i are bounded over the set Z , i.e., there is a scalar $L > 0$ such that for all i

$$\|d\| \leq L \text{ for all } d \in \partial f_i(z) \text{ and all } z \in Z$$

Assumption 7: (*Compactness*) For each i , the local constraint set Z_i is a compact set, i.e., there exists a scalar $B > 0$ such that

$$\|z\| \leq B \text{ for all } z \in Z_i \text{ and all } i$$

The next proposition presents convergence result for the same constraint set case. In particular, it is shown that the iterates of the projected subgradient algorithm (8)-(9) converge to an optimal solution when we use a stepsize converging to zero fast enough.

Proposition 2 [Nedic et al., 2010] *Let Weights Rule, Doubly Stochasticity, Connectivity, Information Exchange, and Same Constraint Set Assumptions hold (cf. Assumptions 2, 3, 4, 5, and 6). Let $\{z_i(k)\}$ be the iterates generated by the algorithm (8)-(9) with the stepsize satisfying $\sum_k \alpha_k = \infty$ and $\sum_k \alpha_k^2 < \infty$. In addition, assume that the optimal solution set Z^* is nonempty. Then, there exists an optimal point $z^* \in Z^*$ such that*

$$\lim_{k \rightarrow \infty} \|z_i(k) - z^*\| = 0 \text{ for all } i$$

The next proposition presents convergence result for the projected subgradient algorithm (8)-(9) in the uniform weight case.

Proposition 3 [Nedic et al., 2010] *Let Interior Point and Compactness Assumptions hold (cf. Assumptions 1 and 7). Let $\{z_i(k)\}$ be the iterates generated by the algorithm (8)-(9) with the weight vectors $a_i(k) = (1/m, \dots, 1/m)'$ for all i and k , and the stepsize satisfying $\sum_k \alpha_k = \infty$ and $\sum_k \alpha_k^2 < \infty$. Then, the sequences $\{z_i(k)\}$, $i = 1, \dots, m$, converge to the same optimal point, i.e.*

$$\lim_{k \rightarrow \infty} \|z_i(k) - z^*\| = 0 \text{ for some } z^* \in X^* \text{ and all } i$$

4.3 Collaborative Multidisciplinary Design Optimization

In this section, we present our results. We consider a distributed multidisciplinary design optimization problem with quadratic objectives and linear constraints. In a special case, we establish conditions under which convergence of the algorithm (17) to optimal solutions is guaranteed. For notational convenience, let w denote

$$w = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Definition 1 The design optimization problem of optimizing a quadratic function of several variables, including (shared) design variables as well as state and coupling variables, subject to linear constraints on these variables is called Quadratic Design Programming (QDP). QDP is a special type of multidisciplinary design optimization problem and can be formulated as follow:

$$\begin{aligned} \min_{w_z} \quad & w^T Q w + P^T w \\ \text{s.t.} \quad & A w \leq b \\ & E w = d \\ & y' = C w \\ & x' = X w \end{aligned} \tag{18}$$

Definition 2 An special class of QDP problem can be defined as follow:

$$\begin{aligned} \min_z \quad & \sum_{i=1}^m c_i(k) (z_i(k) - y_i(k) - t_i(k))^2 \\ & \sum_{r=1}^m (\lambda_r z_r(k) + \mu_r y_r(k)) \leq b \\ \text{s.t.} \quad & y_i(k) = \sum_{r=1}^m b_{ir}(k) z_r(k), \quad i = 1, \dots, m \end{aligned} \tag{19}$$

The next proposition presents convergence result for algorithm (17) for design problem 19.

Proposition 4 *Let the sequence $\{z_i(k)\}$ be generated by the collaborative design optimization algorithm (17) for the QDP problem (19) with the following weights, for all i and j*

$$\begin{aligned} a'_{ij}(k) &= a_{ij}(k) - 2\alpha_i(k)c_j(k)b'_j(k)^2 \\ b'_i(k) &= 1 - \sum_{r=1}^m b_{ri}(k) \end{aligned}$$

Also, let Weights Rule, Doubly Stochasticity, Connectivity, and Information Exchange Assumptions hold (cf. Assumptions 2, 3, 4, and 5) and, let the set $Z = \bigcap_{i=1}^m Z_i$

be nonempty. Under Same Constraint Set Assumptions with the stepsize satisfying $\sum_k \alpha_k = \infty$ and $\sum_k \alpha_k^2 < \infty$. Then, there exists an optimal point $z^* \in Z^*$ such that

$$\lim_{k \rightarrow \infty} \|z_i(k) - z^*\| = 0 \text{ for all } i$$

Proof The proof idea is straightforward. It is to show that the collaborative design optimization algorithm (17) for QDP problem reduces to the distributed constrained optimization equation (2). First, we calculate the gradient of quadratic objective function:

$$\begin{aligned} d'_i(k) &= \nabla_z f(y, z) \\ &= \nabla_z \left[\sum_{i=1}^m c_i(k) (z_i(k) - y_i(k) - t_i(k))^2 \right] \\ &= \nabla_z \left[\sum_{i=1}^m c_i(k) (b'_i(k) z_i(k) - t_i(k))^2 \right] \\ &= \nabla_z \left[\sum_{i=1}^m c_i(k) (b'_i(k) z_i(k) - t_i(k))^2 \right] \\ &= \sum_{i=1}^m c_i(k) \nabla_z (b'_i(k) z_i(k) - t_i(k))^2 \\ &= \sum_{i=1}^m 2c_i(k) b'_i(k) (b'_i(k) z_i(k) - t_i(k)) \end{aligned}$$

where

$$b'_i(k) = 1 - \sum_{r=1}^m b_{ri}(k)$$

By substituting in equation (17)

$$\begin{aligned} z_i(k+1) &= P_{S_i} \left[\sum_{j=1}^m a_{ij}(k) z_j(k) - \alpha_i(k) d'_i(k) \right] \\ &= P_{S_i} \left[\sum_{j=1}^m a_{ij}(k) z_j(k) \cdots \right. \\ &\quad \left. \cdots - \alpha_i(k) \sum_{i=1}^m 2c_i(k) b'_i(k) (b'_i(k) z_i(k) - t_i(k)) \right] \\ &= P_{S_i} \left[\sum_{j=1}^m \left((a_{ij}(k) - 2\alpha_i(k) c_j(k) b'_j(k)^2) z_j(k) \cdots \right. \right. \\ &\quad \left. \left. \cdots - (2\alpha_i(k) c_j(k) b'_j(k)^2) t_j(k) \right) \right] \end{aligned}$$

we conclude

$$z_i(k+1) = P_{S_i} \left[\sum_{j=1}^m \left(a'_{ij}(k) z_j(k) - \beta_{ij}(k) t_j(k) \right) \right]$$

where

$$\begin{aligned} a'_{ij}(k) &= a_{ij}(k) - 2\alpha_i(k) c_j(k) b'_j(k)^2 \\ \beta_{ij}(k) &= 2\alpha_i(k) c_j(k) b'_j(k)^2 \end{aligned}$$

Afterwards, the results are concluded by using proposition (2).

Remark 3 Every distributed design optimization problem without design variables updates (i.e., $\alpha_i(k) = 0$) reduces to a distributed average consensus problem.

Remark 4 It is possible to guarantee convergence of a distributed design optimization problem both through sharing appropriate design variables (i.e., $a_{ij}(k)$) as well as coupling variables (i.e., $b'_j(k)$).

Proposition 5 *Let the sequence $\{z_i(k)\}$ be generated by the collaborative design optimization algorithm (17) for the QDP problem (19) with the following weights, for all i and j*

$$\begin{aligned} a'_{ij}(k) &= a_{ij}(k) - 2\alpha_i(k) c_j(k) b'_j(k)^2 \\ b'_i(k) &= 1 - \sum_{r=1}^m b_{ri}(k) \end{aligned}$$

Also, let Weights Rule, Doubly Stochasticity, Connectivity, and Information Exchange Assumptions hold (cf. Assumptions 2, 3, 4, and 5) and, let the set $Z = \bigcap_{i=1}^m Z_i$ be nonempty. Let the weight vectors

$$a_i(k) = (1/m, \dots, 1/m)' \text{ for all } i \text{ and } k$$

and the stepsize satisfying $\sum_k \alpha_k = \infty$ and $\sum_k \alpha_k^2 < \infty$. Then, the sequences $\{z_i(k)\}$, $i = 1, \dots, m$, converge to the same optimal point, i.e.

$$\lim_{k \rightarrow \infty} \|z_i(k) - z^*\| = 0 \text{ for some } z^* \in Z^* \text{ and all } i$$

Proof Same as proposition (4).

Proposition 6 *Let the sequence $\{z_i(k)\}$ be generated by the collaborative design optimization algorithm (17) for the QDP problem (19) with $t_i(k) = 0$ and the following weights, for all i and j*

$$a'_{ij}(k) = a_{ij}(k) - 2\alpha_i(k)c_j(k)b'_j(k)^2$$

$$b'_i(k) = 1 - \sum_{r=1}^m b_{ri}(k)$$

Also, let *Weights Rule, Doubly Stochasticity, Connectivity, and Information Exchange Assumptions* hold (cf. Assumptions 2, 3, 4, and 5) and, let the set $Z = \bigcap_{i=1}^m Z_i$ be nonempty. We then have for some $\tilde{z} \in Z$ and all i ,

$$\lim_{k \rightarrow \infty} \|z_i(k) - \tilde{z}\| = 0$$

Proof Same as proposition (4). The proof idea is to show that the collaborative design optimization algorithm (17) for QDP problem reduces to the projected consensus algorithm (8)-(9) when $t_i(k) = 0$.

Remark 5 Discussion of another proof of propositions (4), (5), and (6), provided by expanding the projection onto hyperplanes, omitted here due to space limitations (can be found in [Noori, 2012]).

Another special case that may be of interest is Quasi-seperable problems, (i.e., the problems where coupling objectives and constraints are not present). Further discussions of the different special cases will be omitted here in order to reserve more space and time for the discussion of the general problem. The interested reader is referred to [Noori, 2012].

5 PRELIMINARY NUMERICAL RESULTS

The coupling among subspaces and state transitions are the main difference between distributed optimization and design optimization problems (see Figure 3). In fact, guaranteed convergence of the general distributed design optimization problem in the presence of coupling dynamics is difficult. In special cases, it can be shown that this problem can be reduced to the distributed constrained optimization problem. In the following example, we consider an special class of QDP problem defined in (19).

Example 1 Let's consider the following design optimization problem P which is composed of four subproblems P_1, P_2, P_3 and P_4 . The objectives contains both local and shared design variables, linked through coupling variables. For compliance with the definition of QDP problem, subproblems are defined according to (18). The structure of the problem is illustrated in Figure 5.

$$P_1 : \min_{z_1, z_{s1}, z_{s4}} f_1 = (z_1(t) - y_2(t))^2 + (z_{s1}(t) - 10)^2$$

$$+ (z_{s4}(t) - 10)^2$$

$$s.t. \quad z_1(t) + z_{s1}(t) - z_{s4}(t) \leq 1$$

$$x_1(t) - z_3(t) - z_4(t) = 0$$

$$y_1(t) = x_1(t)$$

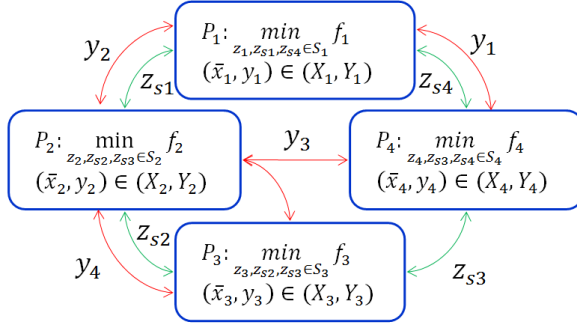


Fig. 5 Example 1

$$\begin{aligned}
 P_2 : \quad & \min_{z_2, z_{s1}, z_{s2}} \quad f_2 = (z_2(t) + y_1(t))^2 + (z_{s1}(t) - 4)^2 \\
 & \quad \quad \quad + (z_{s2}(t) - 4)^2 \\
 \text{s.t.} \quad & z_2(t) - z_{s1}(t) - z_{s2}(t) \leq 1 \\
 & x_2(t) - z_2(t) = 0 \\
 & y_2(t) = x_2(t)
 \end{aligned}$$

$$\begin{aligned}
 P_3 : \quad & \min_{z_3, z_{s2}, z_{s3}} \quad f_3 = (z_3(t) - y_4(t))^2 + (z_{s2}(t) + 2)^2 \\
 & \quad \quad \quad + (z_{s3}(t) - 5)^2 \\
 \text{s.t.} \quad & z_3(t) + z_{s2}(t) + z_{s3}(t) \geq -1 \\
 & x_3(t) - z_{s2}(t) = 0 \\
 & y_3(t) = x_3(t)
 \end{aligned}$$

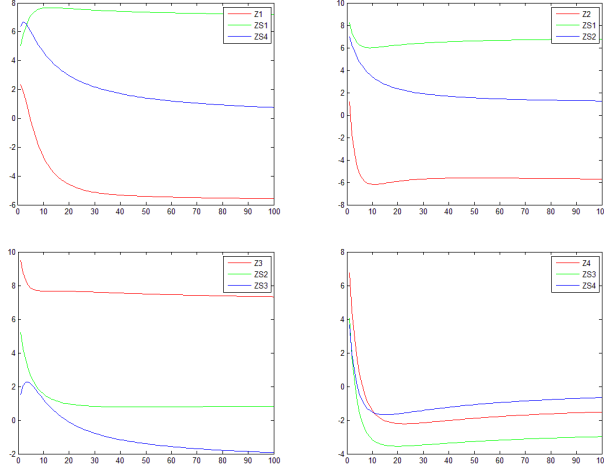
$$\begin{aligned}
 P_4 : \quad & \min_{z_4, z_{s3}, z_{s4}} \quad f_4 = (z_4(t) + y_3(t))^2 + (z_{s3}(t) + 10)^2 \\
 & \quad \quad \quad + (z_{s4}(t) + 10)^2 \\
 \text{s.t.} \quad & z_4(t) - z_{s3}(t) + z_{s4}(t) \geq -1 \\
 & x_4(t) - z_{s1}(t) = 0 \\
 & y_4(t) = x_4(t)
 \end{aligned}$$

Each quadratic objective function are constrained on a one-dimensional locus (i.e., a line) and within a half-space with a hyperplane. The early implementation of the algorithm is done on MATLAB[®] [MathWorks, 2010], in a 32-bit environment. The testbed environment consists of 4 Intel Pentium IV. 2.5GHz and 1GB RAM workstations running Microsoft Windows XP Professional. The PCs are interconnected by a 100Mbit Ethernet LAN setted-up as a single collision domain.

The test results are presented in table 1. The figure 6 shows the performance of the algorithm that uses the following settings; optimization coefficient $\alpha_{opt} = \frac{0.1}{m}$,

Table 1 Comparison of simulation results

Var.	AIO method	CMDO method
z_1	-6.000	-5.999
z_2	-6.000	-6.000
z_3	7.000	7.000
z_4	-0.999	-1.000
z_{s1}	7.000	7.000
z_{s2}	0.999	1.000
z_{s3}	-2.499	-2.499
z_{s4}	0.000	0.000

**Fig. 6** Convergence of the solutions

consensus coefficient $\alpha_{con} = \frac{0.1}{m}$, and iteration $n = 10000$. As depicted in Figure 6, while convergence to the optimal solution is quickly achieved (about 100 iterations) but approaching to accurate result is slow (10000 iterations). This feature of the algorithm could be improved in several ways; amongst them are proper network weight design or proper selection of optimization and consensus factors.

6 CONCLUSION

In this paper, a distributed computation framework for multi-objective multidisciplinary design optimization problems is proposed. The corresponding coordination strategy is collaborative and concurrent, which make it suitable for real-world design problems. It is also shown that distributed constrained optimization problem is an special case of collaborative multidisciplinary design optimization problem. We also investigate an important class of design optimization problems, called QDP problem. By using existing results, we established convergence of an special case of QDP problem. Finally, the paper highlights challenging areas in which research is required to allow us to utilize the full potential of distributed optimization methods in multidisciplinary design optimization in the future.

References

- N. Alexandrov and MY Hussaini, editors. *Multidisciplinary Design Optimization: State of the Art*, Proceedings of the ICASE/NASA Langley Workshop on Multidisciplinary Design Optimization, Hampton, Virginia, March 1995.
- J. K. Allen, S. Azarm, and T. W. Simpson. Designing complex engineered systems. *Journal of Mechanical Design, Transactions of the ASME*, 133, 2011.
- J. T. Allison and P. Y. Papalambros. Consistency constraint allocation in augmented lagrangian coordination. *ASME Journal of Mechanical Design*, 132(7), 2010.
- T.C. Aysal, M.E. Yildiz, A.D. Sarwate, and A. Scaglione. Broadcast gossip algorithms for consensus. *IEEE Transactions on Signal Processing*, 57:2748–2761, 2009.
- V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis. Convergence in multiagent coordination, consensus, and flocking. In *Proceedings of the Joint 44th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC'05)*, 2005.
- V.Y. Blouin, J.B. Lassiter, M.M. Wiecek, and G.M. Fadel. Augmented lagrangian coordination for decomposed design problems. In *Proceedings of the 6th world congress on structural and multidisciplinary optimization*, Rio de Janeiro, 2005.
- S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. *Distributed optimization and statistical learning via the alternating direction method of multipliers*. Now Publishers, 2011.
- R. D. Braun. *Collaborative Optimization: An Architecture for Large-Scale Distributed Design*. PhD thesis, Stanford University, 1996.
- F. Ciucci, T. Honda, and M. C. Yang. An information-passing strategy for achieving pareto optimality in the design of complex systems. *Research in Engineering Design*, 23(1):71–83, 2012.
- E.J. Cramer, Jr. Dennis, P.D. Frank, R.M. Lewis, and G.R. Shubin. Problem formulation for multidisciplinary optimization. *SIAM Journal of Optimization*, 4:754–776, 1994.
- A. V. Demiguel and W. Murray. A local convergence analysis of bilevel decomposition algorithms. *Optimization and Engineering*, 7(2):99–133, 2006.
- C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer-verlag (Graduate Texts in Mathematics), 2001.
- W. M. Haddad, V. S. Chellaboina, and S. G. Nersisov, editors. *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control*. Princeton University Press, 2006.
- R. T. Haftka and L. T. Watson. Multidisciplinary design optimization with quasiseparable subsystems. *Optimization and Engineering*, 6 (1):9–20, 2005.
- G. Hernandez, C. C. Seepersad, and F. Mistree. Designing for maintenance: A game theoretic approach. *Engineering Optimization*, 34:561–577, 2002.
- A. Jadbabaie, J. Lin, and S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transaction on Automatic Control*, 48(6): 988–1001, 2003.
- D. Kempe, A. Dobra, and J. Gehrke. Gossip-based computation of aggregate information. In *Foundations of Computer Science, 2003. Proceedings. 44th Annual IEEE Symposium on*, pages 482–491. IEEE, 2003.
- K. Lewis and F. Mistree. Modeling the interactions in multidisciplinary design: A game theoretic approach. *AIAA Journal of Aircraft*, 35(8):1387–1392, 1997.
- D. Luenberger. *Optimization by vector space methods*. Wiley, New York, 1969.

The MathWorks. *MATLAB programming fundamentals*. The MathWorks, 2010.

N. Michelena, H. M. Kim, and P. Y. Papalambros. A system partitioning and optimization approach to target cascading. In *In Proceedings of the 12th International Conference on Engineering Design*, 1999.

A. Nedic and A. Ozdaglar. Distributed subgradient methods for multi-agent optimization. *IEEE Transactions on Automatic Control*, 54(1):48–61, 2009.

A. Nedic, A. Ozdaglar, and P. A. Parrilo. Constrained consensus and optimization in multi-agent networks. *IEEE Transactions on Automatic Control*, 55(4):922–938, 2010.

A. Noori. Distributed multidisciplinary design optimization algorithms. *Submitted*, 2012.

R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transaction on Automatic Control*, 49(4):1520–1533, 2004.

R. Olfati-Saber, J.A. Fax, and R.M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, 2007.

B.D. Roth and I.M. Kroo. Enhanced collaborative optimization: a decomposition-based method for multidisciplinary design. In *Proceedings of the ASME design engineering technical conferences*, Brooklyn, New York, 2008.

J. S. Sobieski. A linear decomposition method for large optimization problems. Technical report, NASA TM-83248, 1982.

J. S. Sobieski. Optimization by decomposition: A step from hierarchic to non-hierarchic systems. In *2nd NASA Air Force Symposium on Advances in Multidisciplinary Analysis and Optimization*, 1988.

J. S. Sobieski and R. T. Haftka. Multidisciplinary aerospace design optimization: survey of recent developments. *Structural and Multidisciplinary Optimization*, 14:1–23, 1997.

J. S. Sobieski, T. D. Altus, M. Phillips, and J. R. Sandusky. Bilevel integrated system synthesis for concurrent and distributed processing. *AIAA Journal*, 41:1996–2003, 2003.

S. Tosserams, L. F. P. Etman, Y. Papalambros, and J. E. Rooda. An augmented lagrangian relaxation for analytical target cascading using the alternating direction method of multipliers. *Structural and Multidisciplinary Optimization*, 31(3):176–189, 2006.

S. Tosserams, L.F.P. Etman, and J.E. Rooda. An augmented lagrangian decomposition method for quasiseparable problems in mdo. *Structural and Multidisciplinary Optimization*, 34:211–227, 2007.

S. Tosserams, L.F.P. Etman, and J.E. Rooda. Augmented lagrangian coordination for distributed optimal design in mdo. *International Journal of Numerical Methods in Engineering*, 73(13):1885–1910, 2008.

S. Tosserams, L. F. P. Etman, and J. E. Rooda. A classification of methods for distributed system optimization based on formulation structure. *Structural and Multidisciplinary Optimization*, 39:503–517, 2009.

A. Xiao, S. Zeng, J. K. Allen, D. W. Rosen, and F. Mistree. Collaborative multidisciplinary decision making using game theory and design capability indices. *Research in Engineering Design*, 16(1-2):57–72, 2005.